FORMULA SHEET

$a)$ The solution of the following problem :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(L,t) = 0, \ \forall\, t \\ u(x,0) = f(x) \\ u_t(x,0) = g(x), \ \forall\, x \in [0,L] \end{cases}$$

for a string of length L is

$$u(x,t) = \sum_{n\geq 1} \Big(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t) \Big) \sin\left(\frac{n\pi x}{L}\right),$$

with $\lambda_n = \dfrac{cn\pi}{L}$,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad , \quad n = 1, 2, \ldots \text{and}$$

$$B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots$$

$b)$ The solution of the following problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

for a thin metal bar of length L is

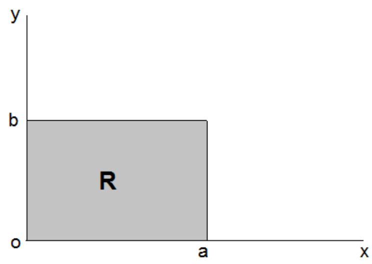
$$u(x,t) = \sum_{n\geq 1} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n{}^2 t},$$

with

$$\lambda_n = \frac{cn\pi}{L}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots$$

$c)$ The solution of the following equation : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in the following rectangle R



such that $u(x,y) = f(x)$ on the upper side and 0 on the other sides of R , is :

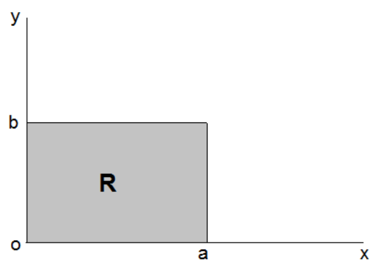
$$u(x,y) = \sum_{n\geq 1} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right),$$

with $B_n = \dfrac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} \displaystyle\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx,$ n=1,2,. . .

d) The solution of the following problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ u = 0 \text{ on the boundary} \\ u(x,y,0) = f(x,y) \\ u_t(x,y,0) = g(x,y) \end{cases}$$

considered in the following rectangular membrane :



is $u(x,y,t) = \sum_{m\geq 1}\sum_{n\geq 1}\Big(B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\Big)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$,

with

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy$$

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy$$

and $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}, m = 1,2,\ldots$ and $n = 1,2,\ldots$

Laplace transform

$a)$ $L\{1\} = \frac{1}{s}$

$b)$ $L\{t^n\} = \frac{n!}{s^{n+1}}$, $n = 1,2,3,\ldots$

$c)$ $L\{e^{at}\} = \frac{1}{s-a}$

$d)$ $L\{\sin(kt)\} = \frac{k}{s^2+k^2}$

$e)$ $L\{\cos(kt)\} = \frac{s}{s^2+k^2}$

$f)$ $L\{f(t-a)U(t-a)\} = e^{-as}F(s)$, where $F(s) = L\{f(t)\}$.

$g)$ If f, f',…,$f^{(n-1)}$ are continuous on $[0,\infty)$ and if $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$, where $F(s) = L\{f(t)\}$.